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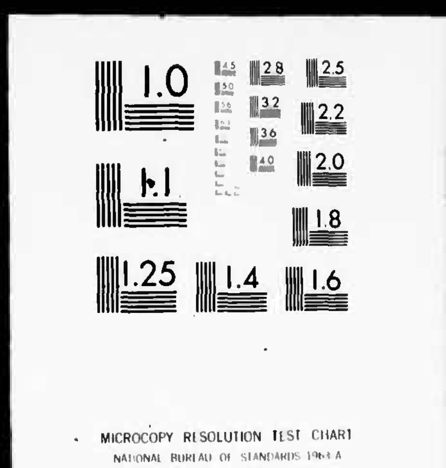
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OPTIMIZED METHOD FOR THE DERIVATION OF THE DEFLECTION OF THE VERTICAL FROM RGSS DATA

prepared for

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I. Introduction

The Rapid Gravity Survey System (RGGS) provides a means of quickly measuring precisely non-astrogeodetic values of the deflections of the vertical. A test vehicle carries an Inertial Positioning System (IPS) which at each of the vehicle's stops produces an error velocity which can be related to the inertial platform tilt errors and the deflections of the vertical. An optimal determination of the gyro drifts and the deflections of the vertical can only be obtained by a post-mission smoothing of the data. In this case, accurate data are available *a priori* for the deflections of the vertical at the start and stop of the vehicle's mission -- as well as the data on the IPS velocity errors at each stop.

The purpose of this report is to develop the equations for the position, velocity, and tilt angle errors into a useable algorithm for the optimal estimation of the deflections of the vertical. This entails three major analytic tasks that will be presented in the succeeding section: one, the development of an expression for the velocity errors at some time, t_n, in terms of the known initial conditions at time, $t = 0$, and the unknown variables to be estimated -- i.e. the gyro drift biases and the values of ξ_i and η_i, the vertical deflections at previous time intervals; two, the replacement of ξ_i and η_i by a smaller number of values $\hat{\xi}_k$ and $\hat{\eta}_k$ -- the needed values of ξ_i and η_i being produced by statistical collocation from the $\hat{\xi}_k$, $\hat{\eta}_k$ basis set; three, a least squares solution for the gyro biases and the $\hat{\xi}_k$, $\hat{\eta}_k$ that minimizes the

mean square deviation of the velocity errors from their estimates.

This mathematical development allows the production of a machine algorithm for use with actual data. A coded version of the algorithm with explanatory comments will make up the final section of this report.

II. Mathematical Development of the Error Adjustment Technique

The differential equations for the horizontal velocity errors are

$$\frac{du}{dt} = \frac{d\dot{x}}{dt} = -S_E\phi_Z + g\phi_E + g\xi \tag{1}$$

$$u = \frac{dx}{dt} \tag{2}$$

$$\frac{dv}{dt} = \frac{d\dot{y}}{dt} = S_N\phi_Z - g\phi_N + g\eta \tag{3}$$

$$\frac{d\phi_Z}{dt} = \tan\varphi\,\frac{v}{R} - (\Omega_N + \zeta_N \sec^2\varphi)\frac{x}{R} + \omega_N\phi_E - \omega_E\phi_N + \alpha \tag{4}$$

$$\frac{d\phi_N}{dt} = \frac{v}{R} + \omega_E\phi_Z - \omega_Z\phi_E + \beta \tag{5}$$

$$\frac{d\phi_E}{dt} = -\frac{u}{R} + \omega_Z\phi_N - \omega_N\phi_Z + \gamma \tag{6}$$

where the meaning of the symbols is as follows:

x	north position error
u	north velocity error
X	north vehicle position
V_N	north vehicle velocity
S_N	north vehicle acceleration
y	east position error
v	east velocity error
Y	east vehicle position
V_E	east vehicle velocity
S_E	east vehicle acceleration
ϕ_Z	azimuth platform attitude error
ϕ_N	platform tilt error about north axis
ϕ_E	platform tilt error about east axis
φ	geographic latitude
g	normal gravity
R	mean earth radius
Ω	earth's angular rotation rate
Ω_S	Schuler rate $\equiv (g/R)^{1/2}$
ω_N	$\Omega \cos \varphi + V_E/R$ north spatial rate
ω_E	$\equiv -V_N/R$ east spatial rate
ω_Z	$\equiv \Omega \sin \varphi + \tan \varphi \; V_E/R$ vertical spatial rate
α	azimuth axis error angular drift rate

β north axis error angular drift rate

γ east axis error angular drift rate

ξ north deflection of the vertical

η east deflection of the vertical

Except for the terms in (1) and (3) involving S_E and S_N, this set may be solved in a straightforward manner. These restrictions are not really limiting, as we will show below. The reduced set of equations is

$$\frac{du}{dt} = g\phi_E - g\xi \tag{7}$$

$$\frac{dx}{dt} = u \tag{8}$$

$$\frac{dv}{dt} = -g\phi_N + g\eta \tag{9}$$

$$\frac{d\phi_Z}{dt} = \tan\varphi\,\frac{v}{R} - \Omega\cos\varphi\,\frac{x}{R} + \Omega\cos\varphi\,\phi_E + \alpha \tag{10}$$

$$\frac{d\phi_N}{dt} = \frac{v}{R} - \Omega\sin\varphi\,\phi_E + \beta \tag{11}$$

$$\frac{d\phi_E}{dt} = -\frac{u}{R} + \Omega \sin\varphi\, \phi_N - \Omega \cos\varphi\, \phi_z + \gamma, \tag{12}$$

where we have neglected terms in V/R relative to the earth rotation rate, Ω . This is also the rationale for dropping -- at least initially -- the terms containing S_N and S_E which are of order vehicle acceleration/ g relative to the leading terms.

Equations (7) - (12) are of the form

$$\frac{d}{dt}\underset{\sim}{\mu} - \underset{\sim}{A}\,\underset{\sim}{\mu} = \underset{\sim}{\Psi} \tag{13}$$

where A is the coefficient matrix,

$$\underset{\sim}{\mu} = \begin{pmatrix} u \\ v \\ z \\ \phi_Z \\ \phi_N \\ \phi_E \end{pmatrix} \quad \text{and} \quad \underset{\sim}{\Psi} = \begin{pmatrix} -g\xi \\ -g\eta \\ 0 \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Assume that the solution to the homogeneous part of (13) has a functional dependence, $e^{\lambda t}$. Then, the homogeneous part of (13) is

$$(\lambda - \underset{\sim}{A})\,\underset{\sim}{\mu} = 0 \tag{14}$$

a standard eigenvalue problem. The secular equation, $\mathrm{Det}(\lambda-\underset{\sim}{A})=0$ gives, as its roots, the values of λ for which (14) is satisfied. The secular equation is

$$\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & -g \\ 0 & \lambda & 0 & 0 & +g & 0 \\ -1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & -\frac{\tan\varphi}{R} & \Omega\cos\varphi & \lambda & 0 & -\Omega\cos\varphi \\ 0 & -\frac{1}{R} & 0 & 0 & \lambda & \Omega\sin\varphi \\ \frac{1}{R} & 0 & 0 & +\Omega\cos\varphi & -\Omega\sin\varphi & \lambda \end{vmatrix} = 0 \qquad (15a)$$

$$\lambda^6 + \lambda^4(2\Omega_s^2 + \Omega^2) + \lambda^2(\Omega_s^4 + \Omega_s^2\Omega^2\sin^2\varphi) - \Omega^2\Omega_s^4\cos^2\varphi = 0 \qquad (15b)$$

which has roots

$$\lambda = \pm i\Omega_s,\ \pm i\Omega',\pm K$$

where $\Omega_s^2 = (\frac{g}{R})^{1/2}$ the Schuler frequency,

$$\Omega' = [\frac{1}{2}(\Omega_s^2 + \Omega^2) + \frac{1}{2}\{\Omega_s^4 + 2\Omega_s^2\Omega^2(1 + 2\cos^2\varphi) + \Omega^4\}^{1/2}]^{1/2} \qquad (16)$$

$$\simeq \{\Omega_s^2 + \Omega^2(1 + \cos^2\varphi)\}^{1/2}$$

and

$$K = [\frac{1}{2}\{\Omega_s^4 + 2\Omega_s^2\Omega^2(1 + 2\cos^2\varphi) + \Omega^4\}^{1/2} - \frac{1}{2}(\Omega_s^2 + \Omega^2)]^{1/2} \qquad (17)$$

$$\simeq \Omega\cos\varphi\,.$$

The eigenvectors may be found by solving (14) for all the other elements of μ in terms of one element, using a specific eigenvalue. The eigenvectors are

$$\underset{\sim}{\mu}^{(i)} = \begin{bmatrix} \frac{g}{\lambda_i} (\lambda_i^2 + \Omega_s^2) \\ g \ \Omega \sin \varphi \\ \frac{g}{\lambda_i^2} (\lambda_i^2 + \Omega_i^2) \\ -\{(\lambda_i^2 + \Omega_i^2)^2 + \lambda_i^2 \Omega^2 \sin \varphi\} / \lambda_i \Omega \cos \varphi \\ -\lambda_i \Omega \sin \varphi \\ \lambda_i^2 + \Omega_s^2 \end{bmatrix} \tag{18}$$

$\underset{\sim}{\mu}^{(i)}$ refers to the eigenvector associated with the i - th eigenvalue. It is also convenient to have the eigenvectors of the adjoint problem, i.e.,

$$\underset{\sim}{\mu}^T (\lambda - \underset{\sim}{A}) = 0$$

which has the same eigenvalues as (14). The eigenvectors here are

$$\begin{aligned} \underset{\sim}{\mu}^{T(i)} = \Big[\, & a_i^{-1} \ [(\lambda_i^2 + \Omega_s^2) (\Omega^2 \cos^2 \varphi - \lambda_i^2)/R, \\ & 2\lambda_i^3 \ \Omega \sin \varphi / R, \\ & \Omega^2 \cos^2 \varphi \lambda_i \ (\lambda_i^2 + \Omega_s^2)/R, \\ & -\Omega \cos \varphi \ \lambda_i^2 \ (\lambda_i^2 + \Omega_s^2) \\ & \Omega \sin \varphi \ \lambda_i^2 \ (\lambda_i^2 - \Omega_s^2) \end{aligned} \tag{19a}$$

$$\lambda_i^3 (\lambda_i^2 + \Omega_i^2)\Big]$$

where

$$a_i = 4\lambda_i^3\Omega^2\Omega_s^2 \sin^2\varphi + 2(\lambda_i^2 + \Omega_s^2) (\lambda_i^4 + \Omega^2\Omega_s^2 \cos^2\varphi)/\lambda_i . \tag{19b}$$

Note that $\underset{\sim}{\mu}^{T(i)}$ is a row vector, and the μ's are normalized such that $\underset{\sim}{\mu}^{T(i)} \cdot \underset{\sim}{\mu}^{(j)} = \delta_{ij}$. We now diagonalize (13) by applying the transformation

$$\underset{\sim}{q} = \underset{\sim}{S}\underset{\sim}{\mu} , \quad q_i = S_{ij} \mu_j$$

where $S_{ij} = \mu_j^{T(i)}$ and $S_{ij}^{-1} = \mu_i^{(j)}$ since $\mu_j^{T(i)} \mu_j^{(k)} = \delta_{ik}$.

Equation (13) becomes

$$\frac{dqi}{dt} - \lambda_i q_i = r_i \tag{20}$$

where $\underset{\sim}{r} = \underset{\sim}{S}\underset{\sim}{Y}$ with a solution, $q_i = c_i e^{\lambda_i t} - r_i/\lambda_i$ (21)

where c_i is a constant. The boundary conditions at $t = 0$ imply $c_i = q_i\ (t=0) + r_i/\lambda_i$, and

$$q_i(t) = q_i(t=0)\ e^{\lambda_i t} + r_i/\lambda_i\ (e^{\lambda_i t} - 1) \tag{22}$$

To get back into the original basis, we apply the inverse transform to (22) and have

$$\begin{aligned} \mu_i(t) &= S_{ij}^{-1} (e^{\lambda_j t} q_j(t=0) + \frac{1}{\lambda_j} (e^{\lambda_j t} -1)r_j) \\ &= S_{ij}^{-1}\ e^{\lambda_j t} S_{jk}\mu_k(t=0) + S_{ij}^{-1} \frac{1}{\lambda_j} (e^{\lambda_j t} -1)S_{jk}\ Y_k , \end{aligned} \tag{23}$$

which completes the solution of the reduced set (7) - (12).

Returning now to the neglected terms involving S_N and S_E, we try a perturbation addition to the solution (23). Thus, for example,

$$u(t) = u_0(t) + u_1(t) + \ldots$$

where u_0 is found from (22), and evidently

$$\frac{du_1}{dt} = -S_E \phi_{zo}(t) \text{ , or}$$

$$u(t) = u_0(t) - \int_0^t S_E \phi_{z_o}(t')dt' \tag{24}$$

Equation (24) could be integrated numerically given the functional form of $S_E(t)$. However, the integration time is expected to be a few minutes (3 - 5) and the typical time scale, $1/\Omega_s$, for variation of the solution is ~ 1/2 hour. This suggests trying a Taylor series for ϕ_{z_0}, or

$$u(t) = u_0(t) - \int_0^t S_E \sum_{n=0}^{\infty} \phi_{z_0}^{(n)} \frac{t'^n}{n!} dt' .$$

Keeping the first two terms in the expansion and integrating by parts gives

$$u(t) = u_0(t) - \phi_{z_0}(t{=}0)V_E \Big|_0^t - \frac{d\phi_{z0}}{dt}(t{=}0) \{V_E t\Big|_0^t - \int_0^t V_E(t')dt'\}$$

$$= u_0(t) + (Y(t) - Y(0)) \frac{d\phi_{z_0}(t{=}0)}{dt} . \tag{25}$$

where Y(t) is the east position of the vehicle and we assume that the vehicle is stopped at time t. If we differentiate and approximate $\frac{d\phi_{z_0}}{dt}$ by $\Omega_s \phi_{z_0}$ since $\frac{d\phi_{z_0}}{dt} \lesssim \Omega_s \phi_{z_0}$, we can compare the size of the two tems in (25), and find

$$\frac{\frac{du_0}{dt}}{\frac{du_1}{dt}} \simeq \frac{g\phi_E}{\Omega_s V_E \phi_z} \simeq \frac{g/R}{\Omega_s V_{E/R}} \simeq \frac{\Omega_s}{V_{E/R}}$$

or the same order as terms thrown out earlier. However, we wish the equations for u and v to be as accurate as possible, since they are the ones that enter directly into the determination of the deflections of the vertical. A further improvement on (25) can be gained by expanding the Taylor series for ϕ_{z_0} about the mid-point of the integration interval. Then, the $\frac{t^2}{2} \frac{\delta^2 \phi_{z0}}{dt^2}$ term will integrate to near zero, because the acceleration of the vehicle will be close to an odd function about the midpoint. In this case (25) is

$$u(t) = u_0(t) + (Y(t) - Y(0)) \{S^{-1}_{\phi_z j} \lambda_j e^{\lambda_j t/2} S_{jk} \mu_k(t=0)$$
$$+ S^{-1}_{\phi_z j} e^{\lambda_j t/2} S_{jk} \Psi_k\} \qquad (26)$$

where the subscript ϕ_z indicates the index required to couple to the variable, ϕ_z. A similar expression can be derived for v(t). The full solution can be written as

$$\mu_i(t) = [S^{-1}_{ij}\, e^{\lambda_j t} S_{jk} + u_{ij} S_{j1}\lambda_1 e^{\lambda_1 t/2} S_{1k}]\mu_k(t=0)$$

$$+ [S^{-1}_{ij}\frac{(e^{\lambda_j t}-1)}{\lambda_j} S_{jk} + u_{ij} S_{j1} e^{\lambda_1 t/2} S_{1k}]\, \Psi_k$$

$$= \Phi_{ik}(t)\mu_k(t=0) + \Lambda_{ik}(t)\, \Psi_k \qquad (27)$$

where $u_{ij} = \begin{cases} 0 & i \neq u,v \text{ and } j \neq \phi_z \\ (Y(t) - Y(0)) & i=u,\ j=\phi_z \\ -(X(t) - X(0)) & i=v,\ j=\phi_z. \end{cases}$

So far we have been concerned with the time evolution of the variables while the vehicle is moving. The vehicle makes frequent stops, however. During these stops the gyro angular errors, ϕ_z, ϕ_N, ϕ_E keep evolving, while the other variables are held constant by fiat. To describe the evolution of angle errors, we extract the relevant equations from (7) - (12) and find

$$\frac{d\phi_z}{dt} = \Omega \cos \varphi\, \phi_E + [\alpha - \Omega \cos\varphi\, \frac{X}{R}] \qquad (28)$$

$$\frac{d\phi_N}{dt} = -\Omega \sin \varphi\ \phi_E + \beta \qquad (29)$$

$$\frac{d\phi_E}{dt} = \Omega \sin\phi_N - \Omega \cos \varphi \, \phi_z + \gamma \qquad (30)$$

where $u = v = 0$ identically and since x is constant, we have lumped it with α in (28). The secular equation of (28) - (30) is

$$\nu \, (\nu^2 + \Omega^2) = 0$$

with roots $\nu = 0$, $\nu = \pm i\Omega$

and transformation matrices

$$\underset{\sim}{S}' = \begin{bmatrix} -\frac{\cos\varphi}{\sqrt{2}} & \frac{\sin\varphi}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{\cos\varphi}{\sqrt{2}} & \frac{\sin\varphi}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \sin\varphi & \cos\varphi & 0 \end{bmatrix} \qquad (31a)$$

and

$$\underset{\sim}{S}'^{-1} = \begin{bmatrix} -\frac{\cos\varphi}{\sqrt{2}} & -\frac{\cos\varphi}{\sqrt{2}} & \sin\varphi \\ \frac{\sin\varphi}{\sqrt{2}} & \frac{\sin\varphi}{\sqrt{2}} & \cos\varphi \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{bmatrix} \qquad (31b)$$

where the rows in $\underset{\sim}{S}'$ and the columns in $\underset{\sim}{S}'^{-1}$ are the eigenvectors corresponding to eigenvalues $i\Omega$, $-i\Omega$, 0, respectively. Referring to (23), we see that the solution of (28) - (30) is

$$\mu_i'(t) = S'^{-1}_{ij} e^{\nu_j t} S'_{jk} \mu_k' (t=0)$$

$$+ S'^{-1}_{ij} \frac{1}{\nu_j} (e^{\nu_j t} - 1) S'_{jk} \Psi_k' \qquad (32)$$

where $\mu_i' = \begin{bmatrix} \phi_Z \\ \phi_N \\ \phi_E \end{bmatrix}$ and $\Psi_i' = \begin{bmatrix} \alpha - \Omega \cos \varphi \frac{x}{R} \\ \beta \\ \gamma \end{bmatrix}$

and $\frac{1}{\nu_j} (e^{\nu_j t} - 1)$ can be understood for $\nu = 0$ as $\lim\limits_{\nu \to 0} = t$.

We rewrite solution (32) in terms of all of the variables, i.e.

$$\mu_i = \Gamma_{ij}(t) \mu_j(t=0) + \Delta_{ij}(t=0) \Psi_j \qquad (33a)$$

with

$$\Gamma_{ij} = \begin{cases} 1 & i = k = u,v,x \\ S'^{-1}_{ij} e^{\nu_j t} S'_{jk} & i,\ k = \phi_Z,\ \phi_N,\ \phi_E \\ S'^{-1}_{ij} \nu_j^{-1} (e^{\nu_j t} - 1)\, S'_{j\phi_z} \left(- \dfrac{\Omega \cos \phi}{R} \right) & i = \phi_Z,\ \phi_N,\ \phi_E \quad k = x \\ 0 & \text{elsewhere} \end{cases} \tag{33b}$$

$$\Delta_{ik} = \begin{cases} S'^{-1}_{ij} \nu_j^{-1} (e^{\nu_j t} - 1)\, S'_{jk} & i,\ k = \phi_Z,\ \phi_N,\ \phi_E \\ 0 & \text{elsewhere} \end{cases} \tag{33c}$$

During the course of time during whihc the vehicle is first stopped for a time $\delta\tau$ and then in motion for a time τ, the evolution of $\underset{\sim}{\mu}$ is given by

$$\mu_i(\tau + \phi\tau) = \Psi_{ij}(\tau)\, \Gamma_{jk}(\delta\tau)\, \mu_k(t=0) + [\Phi_{ij}(\tau)\Delta_{jk}(\delta\tau) + \Lambda_{ik}(\tau)]\, \Psi_k$$

$$= \hat{\Phi}_{ik}(\tau,\delta\tau)\,\mu_k(t=0) + \hat{\Lambda}_{ik}(\tau,\delta\tau)\Psi_k \qquad (34)$$

where $\hat{\Phi}_{ik}$ and $\hat{\Lambda}_{ik}$ are defined by (34). We will refer to $\hat{\Phi}(\tau,\delta\tau)$ and $\hat{\Lambda}(\tau,\delta\tau)$ as time shift operators since they have the effect of updating $\underset{\sim}{\mu}(t)$ to $\underset{\sim}{\mu}(\tau,\tau,\delta\tau)$ for any t.

What we wish to do is to express the observed quantities at some time to the unknown gyro biases, α, β, and γ and unknown vertical deflections. Let us denote the value of $\underset{\sim}{\mu}$ at time t_n as $\underset{\sim}{\mu}^n$, and let $\underset{\sim}{\hat{\Phi}}(\tau_{n-1}, \delta\tau_{n-1}) \equiv \underset{\sim}{\hat{\Phi}}^{n-1}$ where $t_n - t_{n-1} = \tau_{n-1} + \delta\tau_{n-1}$.

Then, for example

$$\underset{\sim}{\mu}^1 = \underset{\sim}{\hat{\Phi}}^{\circ}\underset{\sim}{\mu}^{\circ} + \underset{\sim}{\hat{\Lambda}}^{\circ}\underset{\sim}{\Psi}^{\circ}$$

$$\underset{\sim}{\mu}^2 = \underset{\sim}{\hat{\Phi}}^1\underset{\sim}{\mu}^1 + \underset{\sim}{\hat{\Lambda}}^1\underset{\sim}{\Psi}^1$$

$$= \underset{\sim}{\hat{\Phi}}^1\underset{\sim}{\hat{\Phi}}^{\circ}\underset{\sim}{\mu}^{\circ} + \underset{\sim}{\hat{\Phi}}^1\underset{\sim}{\hat{\Lambda}}^{\circ}\underset{\sim}{\Psi}^{\circ} + \underset{\sim}{\hat{\Lambda}}^1\underset{\sim}{\Psi}^1$$

and in general

$$\underset{\sim}{\mu}^{n+1} = \prod_{i=0}^{n} \underset{\sim}{\hat{\Phi}}^i\,\underset{\sim}{\mu}^{\circ} + \sum_{j=0}^{n-1}\prod_{i=j+1}^{n} \underset{\sim}{\hat{\Phi}}^i\underset{\sim}{\hat{\Lambda}}^j\underset{\sim}{\Psi}^j + \underset{\sim}{\hat{\Lambda}}^n\underset{\sim}{\Psi}^n \qquad (35)$$

In (35) the value of $\underset{\sim}{\mu}$ and, in particular u and v are seen to depend only on the quantities for which we wish to solve and on the

initial boundary conditions. Three further points need to be made. One, the development so far has assumed that the initial values for the succeeding time steps are simply those of the end values of the preceding time step. In fact, however, at each vehicle stop the velocity errors, u and v, are set to zero, and the integration continues with $u = v = 0$ as a starting condition. These boundary conditions can be incorporated by simply replacing the $\hat{\underset{\sim}{\Phi}}$ operator by

$$\bar{\Psi}_{lk} = \hat{\Phi}_{lk}(\delta_{lk} - M_{lk})$$

where

$$M_{lk} = \begin{cases} 0 & l \neq k;\ l,\ k \neq u,\ v \\ 1 & l = k = u, v \end{cases}$$

So finally we have

$$\underset{\sim}{\mu}^{n+1} = \prod_{i=0}^{n} \bar{\underset{\sim}{\Psi}}^{i} \underset{\sim}{\mu}^{*} + \sum_{j=0}^{n-1} \prod_{i=j=0}^{n} \bar{\underset{\sim}{\Psi}}^{i} \hat{\underset{\sim}{\Lambda}}^{j} \underset{\sim}{\Psi}^{j} + \hat{\underset{\sim}{\Lambda}}^{n} \underset{\sim}{\Psi}^{n} \tag{36}$$

Two, the $\underset{\sim}{\Psi}^{n}$ vectors are, in general, a function of time -- at least parametrically through the motion of the vehicle. In order to achieve best accuracy, we should time center the values taken to represent $\underset{\sim}{\Psi}$. Thus, for the gyro drift rates $\alpha^{n-1} = \alpha(t=(t_{n+1}+t_{n})/2)$ gives best accuracy. The vertical deflections do not couple into the evolution of the system while the vehicle is stopped, so that the optimum here is $\xi^{n-1} = \xi(t = t_{n+1} - \tau_{n-1}/2)$. Since the vehicle motion is probably near to being symmetric about this time, we may write

$\xi^{n-1} = \xi([\underset{\sim}{X}^n + \underset{\sim}{X}^{n-1}]/2, [Y^n + Y^{n-1}]/2)$ as a more convenient form of good accuracy. Thus, we use

$$\underset{\sim}{Y}^{n-1} = \begin{bmatrix} \xi([\underset{\sim}{X}^n + \underset{\sim}{X}^{n-1}]/2, [Y^n + Y^{n-1}]/2) \\ \eta([X^n + X^{n-1}]/2, [Y^n + Y^{n-1}]/2) \\ 0 \\ \alpha([t_{n+1} + t_n]/2) \\ \beta([t_{n+1} + t_n]/2) \\ \gamma([t_{n+1} + t_n]/2) \end{bmatrix} \tag{37}$$

as the most accurate representation.

Last, equation (36) is underdetermined. We want an overdetermined system, so as to be able to use a least squares solution for the vertical deflections and gyro biases. Thus, we substitute for ξ^n and η^n by the form

$$\xi^n = \sum_{k=1}^{N} a_{nk} \tilde{\xi}_k + \sum_{k=1}^{N} b_{nk} \tilde{\eta}_k$$

and

$$\eta^n = \sum_{k=1}^{N} c_{nk} \tilde{\xi}_k + \sum_{k=1}^{N} d_{nk} \tilde{\eta}_k$$

The terms with $\tilde{\eta}$ in the determination of ξ, and vice versa, enter because the deflections are strongly cross correlated. The exposition is simplified if we define

$$r_{2i-1} \equiv \xi^{i}$$

$$r_{2i+2} \equiv \eta^{i}$$

then $r_i^{(e)} = \sum_{k=1}^{2N} a_{ik} \tilde{r}_k$

where the (e) superscript denotes an estimate. We would like to minimize the variance of the estimation error, or

$$F \equiv (r_i - r_i^{(e)})^2 = r_i^2 - 2r_i r_i^{(e)} + r_i^{(e)2}$$

$$= r_i^2 - 2r_i \sum_{k=1}^{2N} a_{ik} \tilde{r}_k + \sum_{k=1}^{2N} \sum_{l=1}^{2N} a_{ik} a_{il} \tilde{r}_k \tilde{r}_l$$

To minimize F, we find

$$\frac{\delta F}{\delta a_{ik}} = 0 = -2r_i \tilde{r}_k + 2 \sum_{l=1}^{2N} a_{il} \tilde{r}_k \tilde{r}_l \qquad (38)$$

Taking expectation values of both terms in (38) gives

$$\langle r_i \tilde{r}_k \rangle = \sum_{l=1}^{N} a_{il} \langle \tilde{r}_k \tilde{r}_l \rangle$$

The covariances are matrices and we can solve for a, and have

$$a_{ik} = \langle r_i \tilde{r}_l \rangle \langle \tilde{r}_l \tilde{r}_k \rangle^{-1}$$

where $\langle \tilde{r}_l \tilde{r}_k \rangle^{-1}$ is the inverse matrix of $\langle \tilde{r}_l \tilde{r}_k \rangle$. To specify

$$<r_{2i-1} r_{2i-1}> = <\xi^i \xi^1>$$

$$<r_{2i-1}, r_{2i-2}> = <\xi^i \eta^1> = <\eta^i \xi^1> \qquad (39)$$

$$<r_{2i-2}, r_{2i-2}> = <\eta^i \eta^1>$$

The needed covariances are usually estimated using the assumption of an isotropic, homogeneous covariance. That is the power spectrum of gravity anomaly fluctuations is a function of neither position nor direction. Then the anomaly covariance is a function only of the distance between two points. The covariance of the vertical deflections can be derived given the anomaly covariance (Shaw et al. 1969).

A number of models have been used for the anomaly covariance.

If $\phi_{gg}(r)$ is the anomaly covariance, then

$$\phi_{\xi\xi} = \sigma_\delta^2 \left[\phi_{gg}(r)/\sigma_g^2 + (\sin^2\theta - \cos^2\theta)\, f_c(r)\right]$$

$$\phi_{\eta\eta} = \sigma_\delta^2 \left[\phi_{gg}(r)/\sigma_g^2 + (\cos^2\theta - \sin^2\theta)\, f_c(r)\right]$$

$$\phi_{\xi\eta} = \phi_{\eta\xi} = -2\sigma_\delta^2 \sin\theta \cos\theta\, f_c(r)$$

where σ_g^2 is the anomaly variance, $\sigma_\delta = \sigma_g/g_o \sqrt{2}$, and $r = (X^2 + Y^2)/D$ with D the correlation length. The angle θ is $\tan^{-1}(Y/X)$. The function $f_c(r)$ can be derived once ϕ_{gg} is known. A number of models for the anomaly have been used. For example,

exponential, Bessel, (Shaw et al, 1969) and second order Markovian (Kasper, 1971) models give

exponential

$$\phi_{gg} = \sigma_g^2 \exp(-r)$$

$$f_c(r) = \frac{2}{r^2} - (1 + 2/r + 2/r^2)\, e^{-r}$$

Bessel

$$\phi_{gg} = \sigma_g^2\, r\, K_1(r)$$

$$f_c(r) = 4/r^2 - 2K_o(r) - (4/r + r)\, K_1(r)$$

2nd Order Markovian

$$\phi_{gg} = \sigma_g^2\, e^{-r}\, [1 + r]$$

$$f_c = 6/r^2 - e^{-r}\, [r + 3 + 6/r + 6/r^2]$$

thus, $<\xi^1\eta^1> = -2\sigma_\delta^2 \sin\theta \cos\theta \; f_c(r_{11})$

where $\theta = \tan^{-1} [\, (Y^1 - Y^1)/\, (X^1 - X^1)]$

$r_{11} = \frac{1}{D} [\, (X^1 - X^1)^2 + (Y^1 - Y^1)^2\,]^{1/2}$

and the other elements can be filled similarly.

Rewriting (36) to concentrate on the u^n and v^n variables, we have

$$\begin{matrix} u^n \\ v^n \end{matrix} = \left[\prod_{i=0}^{n-1} \Psi^i \right]_{\substack{u \\ v,k}} \mu_k^0 + \left[\sum_{j=0}^{n-2} \prod_{i=j+1}^{n-1} \Psi^i \hat{\Lambda}^j \right]_{\substack{u \\ v,k}} \Psi_k^j$$

$$+ \hat{\Lambda}^{n-1}_{\substack{u \\ v,k}} \Psi_k^{n-1} \tag{40}$$

where the $[\;]_{ik}$ notation means the i, k'th element of the matrix enclosed in the bracket. It is also convenient to reqrite $\underset{\sim}{\Psi}$ as a sum of vectors, i.e.

$$\underset{\sim}{\Psi}^n = \underset{\sim}{a} + \underset{\sim}{\delta a}^n + \underset{\sim}{b}^n$$

where $a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}$, $\delta a^n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta\alpha^n \\ \delta\beta^n \\ \delta\gamma^n \end{bmatrix}$ and $b^n = \begin{bmatrix} g\xi^n \\ -g\eta^n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The a vector is the mean gyro drift rates over the mission, and the δa vector is the deviations of the drifts from the average. Since we assume

$$\xi^n = r_{2n-1} = \sum_{k=1}^{2N} a_{2n-1,k} \tilde{r}_k$$

$$\eta^n = r_{2n-2} = \sum_{k=1}^{2N} a_{2n-2} \, r_k \;,$$

$$
b^n = \sum_{k=1}^{2N} \begin{pmatrix} \langle r_{2n-1}\tilde{r}_1\rangle \langle \tilde{r}_1\tilde{r}_k\rangle \\ \langle r_{2n-2}\tilde{r}_1\rangle \langle \tilde{r}_1\tilde{r}_k\rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tilde{r}_k
$$

and (40) becomes

$$
\begin{aligned}
\begin{matrix} u^n \\ v^n \end{matrix} - \Big\{ & \Big[\prod_{i=0}^{n-1} \Psi^i \Big]_{\substack{u \\ v},k} \mu \qquad (41) \\
& + \Big[\sum_{j=0}^{n-2} \prod_{i=j+1}^{n-1} \Psi^i \quad \hat{\Lambda}^j + \hat{\Lambda}^{n-1} \Big]_{\substack{u \\ v},k} a_k \\
& + \sum_{j=0}^{n-2} \Big[\prod_{i=j+1}^{n-1} \Psi^i \ \hat{\Lambda}^j \Big]_{\substack{u \\ v}k} \delta a_k^j + \Lambda^{n-1}_{\substack{u \\ v}k} \delta a_k^{n-1} \\
& + \sum_{m=1}^{2N} \Big[\sum_{j=0}^{n-2} \Big\{ \prod_{i=j+1}^{n-1} \Psi^i \ \hat{\Lambda}^j \Big\}_{\substack{u \\ v}k} \begin{pmatrix} \langle r_{2j-1}\tilde{r}_1\rangle\langle r_1 r_m\rangle^{-1} \\ \langle r_{2j-2}\tilde{r}_1\rangle\langle r_1 r_m\rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
& + \hat{\Lambda}^{u-1}_{\substack{u \\ v},k} \begin{pmatrix} \langle r_{2n-3}\tilde{r}_1\rangle\langle \tilde{r}_1\tilde{r}_m\rangle^{-1} \\ \langle r_{2n-4}\tilde{r}_1\rangle\langle \tilde{r}_1\tilde{r}_m\rangle^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Big] \tilde{r}_m \Big\} \equiv F^n_{\substack{u \\ v}}
\end{aligned}
$$

which is in a form suitable for least squares analysis. The terms involving $\delta\alpha$ do not enter into the solution, except to provide an estimate of the weighting of each term of the optimal solution. However, for the current report we shall use even weights and thus drop those terms from the equation. We have then

$$F^{n}_{\substack{u\\v}} = {u^n_{v^n}} - A^{n}_{\substack{u\\v}} \mu^{0}_{k} - B^{n}_{\substack{u\\v},k} a_k - C^{n}_{\substack{u\\v},m} \bar{r}_m \tag{42}$$

where A, B, and C are defined by (32). We want to minimize

$$\sum_{n=1}^{M} \sum_{u,v} F^{n\,2}_{\substack{u\\v}} \equiv G \qquad \text{which implies}$$

$$\frac{\delta G}{\delta a_k} = -2 \sum_{n=1}^{M} \sum_{u,v} F^{n}_{\substack{u\\v}} B^{n}_{\substack{u\\v},k} = 0 \tag{43}$$

$$\frac{\delta G}{\delta \bar{r}_m} = -2 \sum_{n=1}^{M} \sum_{u,v} F^{n}_{\substack{u\\v}} C^{n}_{\substack{u\\v}} \bar{r}_m = 0$$

which gives the following set of simultaneous equations

$$a_l \sum_{n=1}^{M} \sum_{u,v} B^{n}_{\substack{u\\v},l} B^{n}_{\substack{u\\v},k} + \bar{r}_m \sum_{n=1}^{M} \sum_{u,v} C^{n}_{\substack{u\\v},m} B^{n}_{\substack{u\\v},k}$$

$$= \sum_{n=1}^{M} \sum_{u,v} \left({u^n_{v^n}} - A^{u}_{\substack{u\\v}l} \mu^{0}_{l}\right) B^{n}_{\substack{u\\v},k} \qquad \text{permuted over } k \tag{44}$$

and

$$a_1 \sum_{n=1}^{M} \sum_{u,v} B^n_{\substack{u\\v,1}} C^n_{\substack{u\\v,p}} + r_m \sum_{n=1}^{M} \sum_{u,v} C^n_{\substack{u\\v,m}} C^n_{\substack{u\\v,p}} \qquad \text{permuted over } p$$

$$= \sum_{n=1}^{M} \sum_{u,v} \left(u^n_{\substack{\;\\v}\,n} - A^n_{\substack{u\\v,1}} \mu^0_1\right) C^n_{\substack{u\\v,p}} \qquad p \neq 1,2,\ 2N,2N-1$$

which then give the desired gyro rates and deflections of the vertical. The permutation over p is not carried out for $p=1$, 2 or $p=2N$, $2N-1$. These are the values of η and ξ at the start and finish of the mission and are known beforehand.

III. Machine Coded Algorithm

This section presents a FORTRAN source listing of a coded version of the algorithm developed in section II. The program has been debugged to the extent possible without actual test data. Notes on the programming are contained within the comment cards in the listing.

Computer printouts not necessary for understanding of the report.

```
C ************************************************************************
C
C       PROGRAM FOR THE OPTIMIZED, POST-MISSION DETERMINATION OF
C       THE DEFLECTION OF THE VERTICAL USING RGSS DATA
C               PROGRAM PRODUCED BY PHOENIX CORPORATION
C
C ************************************************************************
      COMMON CPHI,SPHI,TPHI
      DIMENSION K(50)
      DIMENSION TGO(51),TSTOP(51),U(51),V(50),XX(51),YY(51),XHAT(51),
     2 YHAT(51),X(50),Y(50),START(6),STEM(6)
      DIMENSION PSI(50,6,6),XL(50,6,6),DUMMY(6,6),DUM2(6,6),DUM3(6,6),
     2DUM4(6,6)
      DIMENSION COEF(100,100),DEFL(100,100),COVAR(100 ,100),
     1 SNMAT(100,100),F(100),VAR(100),SOMAT(100)
      EQUIVALENCE (SUMAT(100,1),SUMAT(1))
      DATA G/9.80665E2/
C
C         INPUT DATA
C            N1 = # OF POINTS AT WHICH A POSITION IS  SPECIFIED
C                 THE FIRST AND LAST POINTS BEING THE START AND STOP
C                 OF THE VEHICLE - THE REMAINDER THE POINTS AT WHICH THE
C                 DEFLECTION IS TO BE DETERMINED.
C            MSTOP = NUMBER OF VEHICLE STOPS IN A GIVEN MISSION
C            TGO(I) = TIME SPENT TRAVELLING ON I-TH LEG
C            TSTOP(I) = TIME SPENT STOPPED ON I-TH LEG
C            U(I) = X OR NRTH VELOCITY ERROR AT END OF I-TH LEG
C            V(I) = EAST OR Y VELOCITY ERROR AT END OF I-TH LEG
C            XX(I) = NORTH POSITION AT END OF I-TH LEG
C            YY(I) = EAST POSITION AT END OF I-TH LEG
C            XHAT,YHAT = NORTH AND EAST POSITIONS AT WHICH THE
C                 DEFLECTION IS TO BE DETERMINED. ' EXCEPTIONS ARE
C                 THE FIRST AND LAST VALUES OF XHAT,YHAT WHICH ARE
C                 THE STARTING PLACE AND FINAL STOPPING PLACE OF THE
C                 MISSION
C            TLAT = TERRESTRIAL LATITUDE OF THE MISSION
C            XIO,ETAO,XIFIN,ETAFIN = THE VALUES OF THE NORTH ANDEAST
C                 DEFLECTIONS OF THE VERTICAL AT THE START AND STOP OF
C                 THE MISSION RESPECTIVELY
C            START(I) = INITIAL CONDITION VECTOR, CONTAINS IN SEQUENCE'
C                 G*XIO,G*ETAO,0.,0.,ETAO,XIO
C
      READ(5,1)        MSTOP,N1
    1 FORMAT(2I5)
      MS = MSTOP + 1
      READ(5,2)(K(I),TGO(I),TSTOP(I),U(I),V(I),XX(I),YY(I),I=1,MS)
    2 FORMAT(I5,6E10.4)
      WRITE(6,101)(I,TGO(I),TSTOP(I),U(I),V(I),XX(I),YY(I),I=1,MS)
  101 FORMAT(1X,I5,6(3X,E10.4))
C
      READ(5,2)(K(I),XHAT(I),YHAT(I),I=1,N1)
      WRITE(6,102)(K(I),XHAT(I),YHAT(I),I=1,N1)
  102 FORMAT(1H1,(I5,2E14.4))
C
      READ(5,3)TLAT,XOO,ETAO,XIFIN,ETAFIN
      READ(5,3)(START(I),I=1,6)
    3 FORMAT(      6E12.4)
```

```
      WRITE(6,103)MSTOP,N1,(START(I),I=1,6),TLAT,XI0,ETA0,XIFIN,ETAFIN
  103 FORMAT(1H0,2I8/2(1X          ,6E14.4/))
        SPHI = SIN(TLAT)
        CPHI = COS(TLAT)
        TPHI = SPHI/CPHI
C
C       INITIALIZE THE TIME ADVANCEMENT OPERATORS
C
      CALL ADVANS
C
C       FIND THE CENTERED VEHICLE POSITION FOR EACH TIME STEP
C
      DO 200 I = 1,MSTOP
      X(I) = (XX(I+1) + XX(I))/2.
  200 Y(I) = (YY(I+1) + YY(I))/2.
C
C       DEFINE THE TIME SHIFT OPERATORS FOR EACH TIME STEP
C                THE PSI OPERATOR GIVES THE CONTRIBUTION COMING FROM
C                THE VALUES OF THE VARIABLES THEMSELVES AT THE
C                BEGINNING OF THE TIME STEP
C                THE XL OPERATOR GIVES THE CONTRIBUTION FROM THE
C                DRIVING TERMS
C
      DO 300 I = 1,MSTOP
      CALL TIME(TGO(I),TSTOP(I),PSI,XL,N,X,Y)
  300 CONTINUE
C
C       DETERMINATION OF THE COEFFICIENT MATRIX OF THE SET OF LINEAR
C       EQUATIONS CONTAINING THE UNKNOWN QUANTITIES.
C           COEF(*,2*N1) CONTAINS THE CONSTANT TERM
C           COEF(*,2*N1-I) I # 3,2,1 CONTAINS THE TERMS INVOLVING
C           COEF(*,J) J # 1 - 2*N1 - 4  CONTAINS THE TERMS INVOLVING
C                ALPHA,BETA,GAMMA, RESP. THE GYRO DRIFT RATES ABOUT
C                VERTICAL,NORTH, EAST AXES, RESP.
C                THE XI : ETA VALUES AT THE CHOSEN POINTS
C
C
C       INITIALIZATION OF THE CONSTANT PART OF THE COEF MATRIX'
C                THE PSI OPERATOR IS SUCCESSIVELY OPERATED ON THE
C                INITIAL CONDITION VECTOR, START, TO PRODUCE THE
C                EFFECT OF THE INITIAL CONDITIONS AT THE START OF THE
C                MISSION ON THE U AND V ERROR VELOCITIES. THE DIFFERENCE
C                BETWEEN THESE QUANTITIES  FORMS PART OF THE TERM INVOLV-
C                BETWEEN THESE QUANTITIES  FORMS PART OF THE CONSTANT
C                TERM, A CONTRIBUTION DUE TO THE DEFLECTION OF THE VERTICA
C                TERM, A CONTRIBUTION DUE TO THE DEFLECTION OF THE
C                OF THE VERTICAL IS SUBTRACTED LATER.
C
      DO 1000 I = 1,MSTOP
      CALL EQUIV(I,PSI,DUMMY)
      DO 1010 KK= 1,6
      STEM(KK)=0.
      DO 1010 L = 1,6
 1010 STEM(KK) = STEM(KK) + DUMMY(KK,L) * START(L)
      DO 1020 L = 1,6
 1020 START(L) = STEM(L)
      COEF(2*I-1,N2) = U(I) - START(1)
```

```
      COEF(2*I,N2) = V(I) - START(2)
 1000 CONTINUE
C
C
C         DETERMINATION OF THE TERMS IN THE COEF MATRIX WHICH DEPEND
C         ON THE  DEFLECTIONS OF THE VERTICAL
C
C         FIRST STEP' DEFINE A MATRIX - DEFL - WHICH GIVES THE VARA
C         FIRST STEP' DEFINE A MATRIX - DEFL - WHICH GIVES THE DEPENDENCE
C                 OF THE VELOCITY ERRORS ON THE VALUES OF THE
C                 DEFLECTIONS AT EACH OF THE MIDPOINTS OF EACH TRAVEL LR
C                 DEFLECTIONS AT EACH OF THE MIDPOINTS OF EACH TRAVEL LEG
C
      DO 1100 I = 1,MSTOP
      CALL EQUIV(I,XL,DUMMY)
      DEFL(2*I-1,2*I-1) = DUMMY(1,1)
      DEFL(2*I-1,2*I) = DUMMY(1,2)
      DEFL(2*I,2*I-1) = DUMMY(2,1)
      DEFL(2*I,2*I) = DUMMY(2,2)
      IF (I.EQ.MSTOP) GO TO 1101
      KK = I + 1
      DO 1110 N = KK,MSTOP
      CALL EQUIV(N,PSI,DUM2)
      CALL XMAT(DUM2,DUMMY,DUM3)
      CALL EQUAL(DUM3,DUMMY)
      DEFL(2*N-1,2*I-1) = DUMMY(1,1)
      DEFL(2*N-1,2*I) = DUMMY(1,2)
      DEFL(2*N,2*I-1) = DUMMY(1,1)
      DEFL(2*N,2*I) = DUMMY(2,2)
 1110 CONTINUE
 1100 CONTINUE
C
C         SECOND STEP' DEFINE THE MATRIX - COVAR - WHICH , WHEN
C                 MULTIPLIED BY DEFL GIVES THE DEPENDENCE OF THE COEF
C                 MATRIX ON THE DEFLECTIONS AT THE DESIRED POINTS
C
      CALL COLLOC(MSTOP,N1,COVAR,X,Y,XHAT,YHAT)
      N2P = N2 - 4
C
C         THIRD STEP' DO THE MULTIPLICATION
C
      DO 1150 I = 1,M2
      DO 1150 J = 1,N2P
      COEF(I,J) = 0.
      DO 1150 L = 1,M2
 1150 COEF(I,J) = COEF(I,J) + DEFL(I,L) * COVAR(L,J+2)
      DO 1160 I = 1,M2
C
C         UPDATE THE CONSTANT PART OF THE COEF MATRIX  TO ACCOUNT FOR THE
C                 KNOWN VALUES OF THE DEFLECTION AT THE START AND STOP
C                 OF THE MISSION
C
      DO 1155 L = 1,M2
      COEF(I,N2) = COEF(I,N2) - G * XI0 * DEFL(I,L) * COVAR(L,1)
     2                        - G * ETA0* DEFL(I,L) * COVAR(L,2)
     3                        - G *XIFIN* DEFL(I,L) * COVAR(L,N2-1)
     4                        - G *ETAFIN*DEFL(I,L)*COVAR(L,N2)
```

```
 1155 CONTINUE
 1160 CONTINUE
C
C
C        FILLING IN THOSE PARTS OF THE COEF MATRIX THAT DEPEND ON THE
C                 GYRO DRIFT RATES
C
      DO 1210 N = 1,3
      DO 1210 I = 1,2
 1210 COEF(I,N2-4+N) = XL(1,I,3+N)
      CALL EQUIV(1,XL,DUMMY)
C
      DO 1200 I = 2,MSTOP
      CALL EQUIV(I,PSI,DUM2)
      CALL EQUIV(I,XL,DUM3)
      CALL XMAT(DUM2,DUMMY,DUM4)
      CALL ADD(DUM4,DUM3,DUMMY)
      DO 1220 N = 1,3
      COEF(2 * I - 1,N2 - 4 + N) = DUMMY(1,3 + N)
 1220 COEF(2*I,N2-4+N) = DUMMY(2,3+N)
 1200 CONTINUE
C
      N2M = N2 - 1
C
C
C        DEFINING THE MATRIX - SQMAT - WHICH SOLVED GIVES THE DESIRED
C                 LEAST SQUARES SOLUTION FOR THE DEFLECTIONS OF THE
C                 VERTICAL AND THE GYRO DRIFT RATES.
C
      DO 1300 I = 1,N2M
      DO 1300 J = 1,N2
      SQMAT(I,J) = 0.
      DO 1300 N = 1,M2
 1300 SQMAT(I,J) = SQMAT(I,J) + COEN(K,I) * COEF(N,J)
C
C        AFTER SUBROUTINE SOLVE, SQMAT(*,2*N) CONTAINS THE SOLUTION
C                 VECTOR' THE LAST  THREE ARE THE GYRO RATES AND THE
C                 REST ARE THE DEFLECTIONS OF THE VERTICAL.
C                 THE LAST COLUMN IS EQUIVALENCED TO A VECTOR SQMAT
C
      CALL SOLVE(SQMAT)
      DO 1400 I = 1,M2
      F(I) = 0.
C
C        DETERMINATION OF THE ACTUAL VARIANCE OF THE SOLUTION
C
      DO 1400 J = 1,N2M
 1400 F(I) = F(I) + COEF(I,J) * SQMAT(J)
      DO 1450 I = 1,M2
      VAR(I) = (F(I) - COEF(I,N2))**2
 1450 SUMSQ = SUMSQ + VAR(I)
      SIGMA = SQRT(SUMSQ/FLOAT(M2-1))
      NN = N2 - 4
      DO 1460 I = 1,NN
      SQMAT(I) = SQMAT(I)/G
C
C
```

```
C         OUTPUT THE FINAL RESULTS
C
      WRITE(6,110) SOMAT(NN+1),SOMAT(NN+2),SOMAT(NN+3)
  110 FORMAT(1H1,' FINAL RESULTS '//'GYRO DRIFT RATES,    ALPHA',E12.4,5X
     2,'BETA',E12.4,5X,'GAMMA',E12.4//'DEFLECTIONS OF VERT'/' XI'
     3',17X,'ETA',17X'NORTH POS',11X,'EAST POS'//)
      WRITE(6,111) (SOMAT(I),SOMAT(I+1),XHAT(I),YHAT(I),I=1,N2,2)
      WRITE(6,112)SUMSQ,SIGMA,(I,VAR(I),I = 1,M2)
  112 FORMAT(1H1,'VARIANCE OF SOLUTION',E12.4,5X,'SIGMA',E12.4/
     ' 'INDIVIDUAL_SQ_ERRORS'/(1X,I5,5X,E12.4))
      STOP
      END
```

```
      SUBROUTINE COLLOC(M,N1,COVAR,X,Y,XHAT,YHAT)
C
C        THIS SUBROUTINE  PRODUCES A MATRIX - COVAR - THAT PRODUCES
C                 VALUES FOR THE DEFLECTION OF THE VERTICAL AT POINTS, I,
C                 FROM THE VALUES OF THE DEFLECTIONS AT OTHER POINTS, J.
C        THIS IS DONE BY STATISTICAL COLLOCATION.
C        FOR A DERIVATION OF THE METHOD SEE THE PHOENIX CORP. REPORT
C
      DIMENSION CV1(50,50),CVINV(50,50),CV2(50,50),COVAR(50,50),
     2  X(50),Y(50),XHAT(50),YHAT(50)
      SIGG2 = SIGMA(1.)
      SIGD2 = SIGD (1.)
C
C        DETERMINE THE COVARIANCES BETWEEN THE DEFLECTIONS AT THE
C                 BASIS POINTS.  ODD INICES  DENOTE XI VALUES, EVEN
C                 INDICES  ETA VALUES. . THE COVARIANCES ARE DERIVED
C                 UNDER THE ASSUMPTION OF ISOTROPIC, HOMOGENEOUS
C                 COVARIANCE OF THE GRAVITY ANOMALY.
C
      DO 500 I = 1,N1
      DO 500 J = 1,N1
      R = SQRT((XHAT(I) -XHAT(JEE**2 + (YHAT(I) - YHAT(J))**2)
      STH = (YHAT(I) - YHAT(J))/R
      CTH = (XHAT(I) - XHAT(J))/R
      CV1(2*I-1,2*J-1) = SIGD2 * (PHIGG(R)/SIGG2 +(STH**2-CTH**2)*FC(R))
      CV1(2*I,2*J) = SIGD2*(PHIGG(R)/SIGG2+(CTH**2- STH**2)*FC(R))
      CV1(2*I-1,2*J)=      -2. * SIGD2 * STH * CTH * FC(R)
      CV1(2*I,2*J-1) = CV1(2*I-1,2*J)
  500 CONTINUE
C
C
C                 INVERSION OF THE CV1 MATRIX  IS FOUND IN CVINV
C
      CALL MATINV(CV1,CVINV)
      DO 600 K = 1,M
C
C        DETERMINATION OF THE COVARIANCES BETWEEN THE BASIS SET AND
C                 THE SET OF POINTS DETERMINED BY THE MISSION LEGS.
C
      DO 600 I = 1,N1
      R = SQRT((X(K) - XHAT(I))**2 + (Y(K)-YHAT(I))**2)
      STH =(Y(K) - YHAT(I))/R
      CTH = (X(K) - XHAT(I))/R
      CV2(2*K-1,2*I-1) = SIGD2 *(PHIGG(R)/SIGG2 +(STH**2- CTH**2)*FC(R))
      CV2(2*K-1,I*1) = -2.*SIGD2*STH*CTH*FC(R)
      CV2(2*K,I*I-1) = CV2(2*K-1,2*I)
  600 CONTINUE
      N2 = 2*N1
      M2 = 2*M
C
C        PRODUCTION OF THE COVAR MATRIX BY MULTIPLICATION OF CV2 BY
C                 CVINV
C
      DO 700 L = 1,M2
      DO 700 K = 1,N2
      COVAR(L,K) = 0.
      DO 700 I = 1,N2
```

```
  700 COVAR(L,K) = COVAR(L,K) + CV2(L,I)*CVINV(I,K)
      RETURN
      END
      FUNCTION SIGMA(X)
C
C         THIS FUNCTION AND ITS ENTRIES GIVE THE NECESSARY VALUES FOR
C                 THE COMPUTATION  OF THE DEFLECTION COVARIANCES.
C         CURRENTLY, THE FUNCTION ASSUMES A SECOND ORDER MARKOVIAN
C                 STRUCTURE FOR THE ANOMALY COVARIANCE.  VAR IS THE
C                 VARIANCE OF THE ANOMALY  AND D IS THE CORRELATION
C                 LENGTH
C
      DATA VAR,D,G,ROOT2/3.36E-2,4E6,9.80665E2,1.41421/
      SIGMA = VAR
      RETURN
      ENTRY SIGD(X)
      SIGD = VAR/G/ROOT2
      RETURN
      ENTRY PHIGG(R)
      Q = R/D
      PHIGG = VAR*EXP(-Q)*(1.+Q)
      RETURN
      ENTRY FC(R)
      Q = R/D
      FC = 6./Q**2 - EXP(-Q) * (Q+3. + 6./Q**2)
      RETURN
      END
      SUBROUTINE ADVANS
C
C         THIS SUBROUTINE PROVIDES THE VALUES OF THE NECESSARY TIME
C                 SHIFT MATRICES.
C         THIS ENTRY INITIALIZES THE VALUES OF THE NECESSARY
C                 EIGENVECTOR MATRICES
C
      COMMON CPHI,SPHI,TPHI
      COMPLEX R(6),P(2),C(6),C2(6),D(2),D2(2),S(6,6),SINV(6,6),T(3,3),
     2 TINV(3,3),PH(6,6),PHI(6,6),VL(6,6),VLI(6,6),PU(3,3),VP(3,3)
      DIMENSION RPHI(6,6),RVLI(6,6),RPU(3,3),RVP(3,3),RPHO(6,6),
     2 DUMMY(6,6),DUM2(6,6),PSI(50,6,6),XL(50,6,6)
      DATA G,REARTH,OMEGA/9.80665E2,6.37103E8,7.29212E-5/
      OMEGAS = SQRT(G/REARTH)
      DO 200 I = 1,6
      DO 200 J = 1,6
  200  RPHO(I,J) = 0.
      DO 210 I = 1,3
  210 RPHO(I,I) = 1.
C
C         FIRST FOR THE TIME WHEN THE VEHICLE IS IN MOTION
C
C
C
C         R CONTAINS THE ROOTS OF THE SECULAR EQUATION
C
      R(1) = (0.,1.) * OMEGAS
      R(2) =-R(1)
      A1 = OMEGA**2 + OMEGAS**2
      B1 = SQRT(OMEGA**4 +OMEGAS**4+2.*OMEGA**2*OMEGAS**2*(1.+2.*CPHI2))
      R(3) = (0.,1)*SQRT(0.5 * A1 + 0.5 * B1)
```

```
      R(4) = - R(3)
      R(5) = (1.,0.)*SQRT(0.5*B1-0.5*A1)
      R(6) = -R(5)
C
C         FILL IN THE VALUES FOR THE TRANSFORMATION MATRIX, S, AND
C                 ITS INVERSE, SINV
C
      DO 500 I = 1,6
      A = (R(I)**2 + OMEGAS**2)
      B=2./R(I)*A**2*(R(I)**4 + (OMEGA*OMEGAS*CPHI)**2)
     2 + 4. * R(I)**3 * (OMEGA*OMEGAS*SPHI)**2
      S(I,1) = A/B/REARTH*((OMEGA*CPHI)**2 - R(I)**2 )
      S(I,2) = 2.*R(I)**3*OMEGA*SPHI/B/REARTH
      S(I,3) = (OMEGA*CPHI)**2 * A* R(I)/B/REARTH
      S(I,4) = -OMEGA * CPHI * R(I)**2*A/B
      S(I,5) = OMEGA*SPHI*R(I)**2*(R(I)**2-OMEGAS**2)/B
      S(I,6) = A/B * R(I)**3
      SINV(1,I) = G*A/R(I)
      SINV(2,I) = G*OMEGA*SPHI
      SINV(3,I) = G*A/R(I)**2
      SINV(4,I) = -(A**2 + (R(I)*OMEGA*SPHI)**2)/(R(I)*OMEGA*CPHI)
      SINV(5,I) = -R(I) * OMEGA*SPHI
  500 SINV(6,I) = A
C
C         NOW DO THE SAME FOR WHEN THE VEHICLE IS STOPPED
C
      P(1) = (0.,1.)* OMEGA
      P(2) = -P(1)
      A1 = 1./SQRT(2.)
      T(1,1) = -CPHI * A1
      T(1,2) = SPHI * A1
      T(1,3) = (0.,1.)*A1
      T(2,1) = T(1,1)
      T(2,2) = T(1,2)
      T(2,3) = - T(1,3)
      T(3,1) = SPHI
      T(3,2) = CPHI
      T(3,3) = 0.
      TINV(1,1) = -CPHI * A1
      TINV(1,2) = T(1,1)
      TINV(1,3) = SPHI
      TINV(2,1) = SPHI * A1
      TINV(2,2) = TINV(2,1)
      TINV(2,3) = CPHI
      TINV(3,1) = -(0.,1.) * A1
      TINV(3,2) = -TINV(3,1)
      TINV(3,3) = 0
      ENTRY TIME(T1,T2,PSI,XL,N,XPOS,YPOS)
C
C         THIS ENTRY ACTUALLY CALCULATES THE TIME SHIFT MATRICES
C                 T1 IS THE TIME THE VEHICLE IS MOVING
C                 T2 IS THE TIME STOPPED
C                 THE DERIVATION OF THE VARIOUS OPERATIONS PERFORMED
C                 HERE IS FOUND IN THE PHOENIX CORP. REPORT
C
      T = T1 * 0.5
      DO 600 K = 1,6
```

```
      C(L) = CEXP(R(K)*T)
  600 C2(K) = C(L) * C(K)
      DO 700 I = 1,6
      DO 700 J = 1,6
      PH(I,J) = (0.,0.)
      PHI(I,J) =(0.,0.)
      VL(I,J) = (0.,0.)
      VLI(I,J) = (0.,0.)
      DO 700 K = 1,6
      PHI(I,J) = PHI(I,J) + C2(K)*SINV(I,K)*S(K,J)
      PH(I,J) = PH(I,J) + R(K) * C(K)*SINV(I,K)*S(K,J)
```

```
      VLI(I,J) = VLI(I,J) + (C2(K)-(1.,0.))/R(K)*SINV(I,K)+S(K,J)
  700 VL(I,J) = VL(I,J) + ((K)*SINV(I,K)*S(K,J))
      DO 800 I = 1,6
      PHI(1,I) = PHI(1,I) + PH(4,I)*(YPOS(N+1)-YPOS(N))
      PHI(2,I) = PHI(2,I) - PH(4,I)*(XPOS(N+1)-XPOS(N))
      VLI(1,I) = VLI(1,I) + VL(4,I) * (YPOS(N+1) - YPOS(N))
  800 VLI(2,I) = VLI(2,I) - VL(4,I) * (XPOS(N+1) - XPOS(N))
      DO 810 I = 1,6
      DO 810 J=1,6
      RPHI(I,J) = REAL(PHI(I,J))
  810 RVLI(I,J) = REAL(VLI(I,J))
      D(1) = CEXP(P(1) * T2)
      D(2)=CEXP(P(2) * T2)
      DO 900 I = 1,3
      DO 900 J = 1,3
      PO(I,J) = D(1) * TINV(I,1)*T(1,J) + D(2) * TINV(I,2)*T(2,J)
     2    + TINV(I,3) * T(3,J)
  900 VP(I,J) = (D(1)-1.)/P(1)*TINV(I,1)*T(1,J)
     2 +(D(2)-1.) / P(2) * TINV(I,2) * T(2,J) + T2 * TINV(I,3) * T(3,J)
      DO 910 I = 1,3
      DO 910 J = 1,3
      RPO(I,J) = REAL(PO(I,J))
  910 RVP(I,J) = REAL(VP(I,J))
      DO 920 I = 1,3
      RPHO(3+I,3) = -OMEGA*CPHI/REARTH*RVPI(I,1)
      DO 920 J = 1,3
      RVPO(3+I,3+J) = RVP(I,J)
  920 RPHO(3+I,3+J) = RPO(I,J)
      CALL XMAT(RPHI,RPHO,DUMMY)
      CALL EQUI(N,PSI,DUMMY)
      DO 950 I = 1,6
      DO 950 J = 1,2
  950 PSI(N,I,J) = 0.
      CALL XMAT(RPHI,RVPO,DUMMY)
      CALL ADD(RVLI,DUMMY,DUM2)
      CALL EQUI(N,XL,DUM2)
      RETURN
      END
```

```
C
C        HANDY MATRIX OPERATIONS NOT FOUND IN FORTRAN
C
C        EQUIV  MAKES A SMALL SEC
C        EQUIV  SETS A SMALL 6X6 MATRIX EQUAL TO A SECTION A LARGER
C                 ONE, EQUI IS THE INVERSE
C        ADD ADDS TWO MATRICES, XMAT MULTIPLIES TWO MATRICES AND
C        EQUAL SETS ONE MATRIX EQUAL TO ANOTHER
C
      SUBROUTINE EQUIV(N,A,B)
      DIMENSION A(50,6,6),B(6,6)
      DO 200 I = 1,6
      DO 200 J = 1,6
  200 B(I,J) = A(N,I,J)
      RETURN
      END
      SUBROUTINE XMAT(A,B,C)
      DIMENSION A(6,6),B(6,6),C(6,6)
      DO 200 I = 1,6
      DO 200 J = 1,6
      C(I,J) = 0.
      DO 200 K = 1,6
  200 C(I,J) = C(I,J) + A(I,K) * B(K,J)
      RETURN
      END

      SUBROUTINE EQUAL(A,B)
      DIMENSION A(6,6),B(6,6)
      DO 200 I = 1,6
      DO 200 J = 1,6
  200 B(I,J) = A(I,J)
      RETURN
      END

      SUBROUTINE ADD(A,B,C)
      DIMENSION A(6,6),B(6,6),C(6,6)
      DO 200 I = 1,6
      DO 200 J = 1,6
  200 C(I,J) = A(I,J) + B(I,J)
      RETURN
      END

      SUBROUTINE EQUI(N,A,B)
      DIMENSION A(50,6,6),B(6,6)
      DO 200 I = 1,6
      DO 200 J = 1,6
  200 A(N,I,J) = B(I,J)
      RETURN
      END
```

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Rapid Gravity Survey System (RGSS) provides a means of quickly measuring precisely non-astrogeodetic values of the deflections of the vertical. A test vehicle carries an Inertial Positioning System (IPS) which at each of the vehicle's stops produces an error velocity which can be related to the inertial platform tilt errors and the deflections of the vertical. An optimal determination of the gyro drifts and the deflections of the vertical can only be obtained by a post-mission smoothing of the data. In this case, accurate data are available a priori for the deflections of the vertical at the start and stop of the vehicle's → Next page

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Block 20. Continued.

mission -- as well as the data on the IPS velocity errors at each stop.

The purpose of this report is to develop the equations for the position, velocity, and tilt angle errors into a useable algorithm for the optimal estimation of the deflections of the vertical. As a result, three major analytic tasks are presented. This mathematical development enables the production of a machine algorithm for use with actual data. The final section of the report contains a coded version of the algorithm with explanatory comments.

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